Queueing for ergodic arrivals and services

László Györfi * Gusztáv Morvai †

February 1, 2008

Abstract

In this paper we revisit the results of Loynes (1962) on stability of queues for ergodic arrivals and services, and show examples when the arrivals are bounded and ergodic, the service rate is constant, and under stability the limit distribution has larger than exponential tail.

*Dept. of Computer Science and Information Theory, Technical University of Budapest, 1521 Stoczek u. 2, Budapest, Hungary. E-mail:gyorfi@szit.bme.hu

†Research Group for Informatics and Electronics of the Hungarian Academy of Sciences, 1521 Stoczek u. 2, Budapest, Hungary. E-mail: morvai@szit.bme.hu

1 Introduction

The analysis of a queueing model consists of two steps: stability study and the characterization of the limit distribution. In this paper we consider long range dependence, i.e. assume ergodic arrivals and services. Concerning stability revisit the result of Loynes (1962), who extended the Markovian approach in an elegant way. For the weak dependent situation the limit distribution usually has an almost exponential tail. Here we show counterexample for ergodic situation, i.e. when the arrivals are bounded and ergodic, the service rate is constant.

2 Stability

2.1 General result

Let X_0 be arbitrary random variable. Define

$$X_{n+1} = (X_n + Z_{n+1})^+ \tag{1}$$

for $n \geq 0$, where $\{Z_i\}$ is a sequence of random variables. We are ineterested in the stability of $\{X_i\}$, i.e. we are looking for conditions on $\{Z_i\}$ under which X_i has a limit distribution. For the classical Markovian approach $\{Z_i\}$ are independent and identically distributed, when stability of $\{X_i\}$ means that there exists a unique limit distribution of X_n. Here consider the case when $\{Z_i\}$ is only stationary and ergodic.

Following Lindvall (1992) introduce a stronger concept of stability:

Definition 1 *We say that the sequence $\{X_i\}$ is coupled with the sequence $\{X'_i\}$ if $\{X'_i\}$ is stationary and ergodic and there is an almost surely finite random variable τ such that*

$$X'_n = X_n$$

for $n > \tau$.

Put

$$\begin{aligned} V_0 &= 0, \\ V_n &= \sum_{i=0}^{n-1} Z_{-i},\ (n \geq 1). \end{aligned}$$

Theorem 1 *If $\{Z_i\}$ is stationary and ergodic, and $\mathbf{E}\{Z_i\} < 0$, then $\{X_i\}$ is coupled with a stationary and ergodic $\{X'_i\}$ such that*

$$X'_0 = \sup_{n \geq 0} V_n.$$

Proof.
Step 1. Let $X_{-N,-N} = 0$ and define $X_{-N,n}$ for $n > -N$ by the following recursion,

$$X_{-N,n+1} = (X_{-N,n} + Z_{n+1})^+ \text{ for } n \geq -N.$$

We show that $X_{-N,0}$ is monoton increasing in N, and almost surely,

$$\lim_{N\to\infty} X_{-N,0} = X',$$

where

$$X' = \sup_{n\geq 0} V_n,$$

and X' is finite a.s.
Notice that $X_{-N,n+1} = (X_{-N,n} + Z_{n+1})^+$ for $n \geq -N$. First we proove that for $n > -N$,

$$X_{-N,-n} = \max\{0, Z_n, Z_n + Z_{n-1}, \ldots, Z_n + \ldots + Z_{-N+1}\}. \tag{2}$$

For $n = -N + 1$,

$$\begin{aligned} X_{-N,-N+1} &= (X_{-N,-N} + Z_{-N+1})^+ \\ &= (Z_{-N+1})^+ \\ &= \max\{0, Z_{-N+1}\}. \end{aligned}$$

For $n = -N + 2$,

$$\begin{aligned} X_{-N,-N+2} &= (X_{-N,-N+1} + Z_{-N+2})^+ \\ &= \max\{0, X_{-N,-N+1} + Z_{-N+2}\} \\ &= \max\{0, \max(0, Z_{-N+1}) + Z_{-N+2}\} \\ &= \max\{0, Z_{-N+2}, Z_{-N+2} + Z_{-N+1}\}. \end{aligned}$$

Now we proove by induction from n to $n + 1$.

$$\begin{aligned} X_{-N,n+1} &= (X_{-N,n} + Z_{n+1})^+ \\ &= \max\{0, \max\{0, Z_n, Z_n + Z_{n-1}, \ldots, Z_n + \ldots + Z_{-N+1}\} + Z_{n+1}\} \\ &= \max\{0, Z_{n+1}, Z_{n+1} + Z_n, \ldots, Z_{n+1} + \ldots + Z_{-N+1}\}. \end{aligned}$$

We have completed the proof of (2). Thus

$$X_{-N,0} = \max\{0, Z_0, Z_0 + Z_{-1}, \ldots, Z_0 + \ldots + Z_{-N+1}\},$$

which imlies that $X_{-N,0}$ is monoton increasing, since the maximum is taken over larger and larger set. It remains to prove that $X_{-N,0}$ converges to a random variable X' which is finite a.s.. Now by the Birkoff strong law of large numbers for ergodic sequences, a.s.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=-N+1}^{0} Z_i = \mathbf{E}Z_1 < 0$$

(cf. Theorem 3.5.7. in Stout [10]), hence a.s.

$$\lim_{N\to\infty} \sum_{i=-N+1}^{0} Z_i = -\infty.$$

We got that there is a random variable τ such that for all $i > 0$

$$\infty > X_{-\tau,0} = X_{-\tau-i,0},$$

and therefore

$$X' = \sup_{n\geq 0} V_n.$$

Step 2. Put

$$X'_0 = X'$$

and for $n \geq 0$,

$$X'_{n+1} = (X'_n + Z_{n+1})^+.$$

We show that $\{X'_i\}$ is stationary and ergodic.
For any sequence $z_{-\infty}^{\infty} = (\dots, z_{-1}, z_0, z_1, \dots)$ put

$$F(z_{-\infty}^{\infty}) = \lim_{N\to\infty} \max\{0, z_0, z_0 + z_{-1}, \dots, z_0 + \dots + z_{-N}\}.$$

Then by Step 1

$$X'_0 = X' = F(Z_{-\infty}^{\infty}).$$

We prove by induction that for $n \geq 0$,

$$X'_n = F(T^n Z_{-\infty}^{\infty}),$$

where T is the left shift. For $n = 1$,

$$\begin{aligned}
F(TZ_{-\infty}^{\infty}) &= \lim_{N\to\infty} \max\{0, Z_1, Z_1 + Z_0, \dots, Z_1 + Z_0 + \dots, Z_{-N+1}\} \\
&= (\lim_{N\to\infty} \max\{0, \max\{0, Z_0, \dots, Z_0 + \dots, Z_{-N+2}\} + Z_1\}) \\
&= (\max\{0, [\lim_{N\to\infty} \max\{0, Z_0, \dots, Z_0 + \dots, Z_{-N+2}\}] + Z_1\}) \\
&= (X'_0 + Z_1)^+ \\
&= X'_1.
\end{aligned}$$

Now we prove from n to $n + 1$.

$$\begin{aligned}
X'_{n+1} &= (X'_n + Z_{n+1})^+ \\
&= (F(T^n Z_{-\infty}^{\infty}) + Z_{n+1})^+ \\
&= (\lim_{N\to\infty} \max\{0, Z_n, Z_n + Z_{n-1}, \dots, Z_n + Z_{n-1} + \dots, Z_{n-N}\} + Z_{n+1})^+ \\
&= \lim_{N\to\infty} \max\{0, Z_{n+1}, Z_{n+1} + Z_n, \dots, Z_{n+1} + Z_n + \dots, Z_{n+1-N}\} \\
&= F(T^{n+1} Z_{-\infty}^{\infty}).
\end{aligned}$$

Step 3. Similarly to the proof of Step 1,

$$X_n = \max\{0, Z_n, Z_n + Z_{n-1}, \dots, Z_n + \dots + Z_1, Z_n + \dots + Z_1 + X_0\},$$

and

$$X'_n = \max\{0, Z_n, Z_n + Z_{n-1}, \ldots, Z_n + \ldots + Z_1, Z_n + \ldots + Z_1 + X'_0\}.$$

But for large n, both

$$Z_n + \ldots + Z_1 + X_0 < 0$$

and

$$Z_n + \ldots + Z_1 + X'_0 < 0,$$

and so

$$X_n = X'_n = \max\{0, Z_n, Z_n + Z_{n-1}, \ldots, Z_n + \ldots + Z_1\}.$$

The proof of Theorem 1 is complete.

Remark 1. From the proof it is clear that the following extension is straightforward: If $\{Z_i\}$ is coupled with a stationary and ergodic $\{Z'_i\}$, and $\mathbf{E}\{Z'_i\} < 0$, then $\{X_i\}$ is coupled with a stationary and ergodic $\{X'_i\}$ such that

$$X'_0 = \sup_{n \geq 0} V_n,$$

where

$$\begin{aligned} V_0 &= 0, \\ V_n &= \sum_{i=0}^{n-1} Z'_{-i}, \ (n \geq 1). \end{aligned}$$

2.2 Queue length for discrete time queueing

As an application of Theorem 1 consider a discrete time queueing with constant service rate s, and denote by Y_n the number of arrivals in time slot n. Let the initial length of the queue Q_0 be arbitrary non-negative integer valued random variable. Then

$$Q_{n+1} = (Q_n - s + Y_{n+1})^+$$

for $n \geq 0$. Put

$$\begin{aligned} V_0 &= 0, \\ V_n &= \sum_{i=0}^{n-1} Y_{-i} - ns, \ (n \geq 1). \end{aligned}$$

Corollary 1 *If $\{Y_i\}$ is stationary and ergodic, and $\mathbf{E}\{Y_i\} < s$, then then $\{Q_i\}$ is coupled with a stationary and ergodic $\{Q'_i\}$ such that*

$$Q'_0 = \sup_{n \geq 0} V_n.$$

Proof. Apply Theorem 1 for

$$Z_n = Y_n - s.$$

Remark 2. This result together with Remark 1 has some consequences for network of servers (tandem of queues), when the output of a server $(Q_n + Y_{n+1} - Q_{n+1})$ is the input of another server. It is easy to show that the output $(Q_n + Y_{n+1} - Q_{n+1})$ is coupled with a stationary and ergodic sequence, and the expectations of the input and the output are equal, so the stability condition holds for the next server, too.

2.3 Wating time for generalized G/G/1

This is another application of Theorem 1. According to Lindley (1952) consider the extension of the G/G/1 model. Let W_n be the waiting time of the n-th arrival, S_n be the service time of the n-th arrival, and T_{n+1} be the inter arrival time between the $(n+1)$-th and n-th arrivals. Let W_0 be an arbitrary random variable. Then

$$W_{n+1} = (W_n - T_{n+1} + S_n)^+$$

for $n \geq 0$. Put

$$\begin{aligned} V_0 &= 0, \\ V_n &= \sum_{i=0}^{n-1} (S_{-i-1} - T_{-i}),\ (n \geq 1). \end{aligned}$$

Corollary 2 *(Extension of Loynes (1964)) If $\{S_{i-1} - T_i\}$ is stationary and ergodic, $\mathbf{E}\{S_{i-1}\} < \mathbf{E}\{T_i\}$, then $\{W_i\}$ is coupled with a stationary and ergodic $\{W'_i\}$ such that*

$$W'_0 = \sup_{n \geq 0} V_n.$$

Proof. Apply Theorem 1 for

$$Z_n = S_{n-1} - T_n.$$

3 Limit distribution

In a queueing problem the properties of the limit distribution are of great importance. In this section we consider the special case of Section 2.2., when the arrivals $\{Y_n\}$ are ergodic and the service rate is constant s. If $\{Y_n\}$ are weakly dependent then the tail of the limit distribution is almost exponential, which may result in efficient algorithms for call admission control (cf. Duffield, Lewis, O'Connel, Russel Toomey (1995)). The exponential tail distribution can be derived using large deviation technique (cf. Glynn, Whitt (1994)). The basic tool in this respect is the cummulant moment generating function:

$$\lambda(\theta) = \lim_{n \to \infty} \frac{1}{n} \log \mathbf{E}\{e^{\theta \sum_{k=1}^{n} Y_k}\},$$

assuming that this limit exists. If the set

$$\{\theta; \lambda(\theta) - \theta s < 0\},$$

is not empty then put

$$\delta = \sup\{\theta; \lambda(\theta) - \theta s < 0\}.$$

Then for large q

$$\mathbf{P}\{Q > q\} \simeq e^{-\delta q}.$$

The question is whether under the stability condition $E(Y_1) < s$ one has exponential tail distribution.

Glynn, Whitt (1994) gave a positive answer under weakly dependent $\{Y_n\}$. The limit in the definition of $\lambda(\theta)$ exists only under some conditions, for example, if $\{Y_n\}$ form a binary Markov chain then $\lambda(\theta)$ can be calculated (Dembo, Zeitouni (1992)), an explicite bound on $\mathbf{P}\{Q > q\}$ can be given (Duffield (1994)).

For a possible extension the other problem is that $\mathbf{E}(Y_1) < s$ is only a necessary condition that the set $\{\theta; \lambda(\theta) - \theta s < 0\}$ is not empty, but not sufficient. This can be seen by Jensen's inequality:

$$\frac{1}{n} \log \mathbf{E}\{e^{\theta \sum_{k=1}^{n} Y_k}\} > \frac{1}{n} \log e^{\theta \mathbf{E}\{\sum_{k=1}^{n} Y_k\}} = \theta \mathbf{E}\{Y_1\},$$

therefore if the set $\{\theta; \lambda(\theta) - \theta s < 0\}$ is not empty then $E(Y_1) < s$.

For long range dependent arrivals Duffield, O'Connel (1995) proved that the tail may not be exponential. They introduced the scaled cummulant moment generating function:

$$\lambda^*(\theta) = \lim_{n\to\infty} \frac{1}{v(n)} \log \mathbf{E}\{e^{\theta \frac{v(n)}{a(n)}[\sum_{k=1}^{n} Y_k - ns]}\},$$

assuming that this limit exists, where $a(t)$ and $v(t)$ are monoton increasing functions. If the set

$$\{\theta; \lambda^*(\theta) < 0\},$$

is not empty then put

$$\delta = \sup\{\theta; \lambda^*(\theta) < 0\}.$$

Then for large q

$$\mathbf{P}\{Q > q\} \simeq e^{-\delta v(a^{-1}(q))}.$$

Duffield, O'Connel (1995) applied this result when $\{\sum_{k=1}^{n} Y_k\}$ is a Gaussian process with stationary increments, or Ornstein-Uhlenbeck process, or a squared Bessel process. These examples can be motivated by multiplexing many sources. For all these examples Y_n is unbounded. In this section we consider bounded Y_n, especially binary valued Y_n. Show examples such that Q has larger tail than exponential.

Proposition 1 *There is a stationary, ergodic and binary valued* $\{Y_n\}$ *such that* $\mathbf{E}\{Y_1\} \leq 1/2$ *and with* $s = 3/4$ *the queue length sequence is stable, and for each* $\delta > 0$ *there is* q *such that*

$$\mathbf{P}\{Q > q\} > e^{-\delta q}.$$

Proof. From Theorem 1

$$Q = \sup_{n \geq 0} V_n,$$

therefore for any n

$$\mathbf{P}\{Q > q\} \geq \mathbf{P}\{V_n > q\} = \mathbf{P}\{\sum_{j=0}^{n-1} Y_{-j} > ns + q\}.$$

We show that for $\delta_i = \frac{1}{i}$, $q_i = 2i^2$, $n_i = 2^{i-1}$ $(i > 16)$,

$$\mathbf{P}(\sum_{j=0}^{n_i-1} Y_{-j} > \frac{3}{4} n_i + q_i) > 2^{-\delta_i q_i}.$$

Step 1. We present first a dynamical system given in Györfi, Morvai, Yakowitz (1998). We will define a transformation T on the unit interval. Consider the binary expansion r_1^∞ of each real-number $r \in [0,1)$, that is, $r = \sum_{i=1}^\infty r_i 2^{-i}$. When there are two expansions, use the representation which contains finitely many $1's$. Now let

$$\tau(r) = \min\{i > 0 : r_i = 1\}. \tag{3}$$

Notice that, aside from the exceptional set $\{0\}$, which has Lebesgue measure zero τ is finite and well-defined on the closed unit interval. The transformation is defined by

$$(Tr)_i = \begin{cases} 1 & \text{if } 0 < i < \tau(r) \\ 0 & \text{if } i = \tau(r) \\ r_i & \text{if } i > \tau(r). \end{cases} \tag{4}$$

Step 2. We show that the transformation T is ergodic.
Notice that in fact, $Tr = r - 2^{-\tau(r)} + \sum_{l=1}^{\tau(r)-1} 2^{-l}$. All iterations T^k of T for $-\infty < k < \infty$ are well defined and invertible with the exeption of the set of dyadic rationals which has Lebesgue measure zero. In the future we will neglect this set. Transformation T could be defined recursively as

$$Tr = \begin{cases} r - 0.5 & \text{if } 0.5 \leq r < 1 \\ \frac{1+T(2r)}{2} & \text{if } 0 \leq r < 0.5. \end{cases}$$

Let

$$S_i = \{I_0^i, \ldots, I_{2^i-1}^i\}$$

be a partition of $[0,1)$ where for each integer j in the range $0 \leq j < 2^i$ I_j^i is defined as the set of numbers $r = \sum_{v=1}^\infty r_v 2^{-v}$ whose binary expansion $0.r_1, r_2, \ldots$ starts with the bit sequence $j_1, j_2, \ldots, j_i$ that is reversing the binary expansion $j_i, \ldots, j_2, j_1$ of the number $j = \sum_{l=1}^i 2^{l-1} j_l$. Observe that in S_i there are 2^i left-semiclosed intervals and each interval I_j^i has length (Lebesgue measure) 2^{-i}. Now I_j^i is mapped linearly, under T onto I_{j-1}^i for

$j = 1, \ldots, 2^i - 1$. To confirm this, observe that for $j = 1, \ldots, 2^i - 1$, if $r \in I_j^i$ then

$$\begin{aligned} Tr &= \sum_{l=1}^{\tau(r)-1} 2^{-l} + \sum_{l=\tau(r)+1}^{\infty} r_l 2^{-l} \\ &= r - \sum_{l=1}^{i} 2^{-l}(j_l - (j-1)_l) \\ &= \sum_{l=1}^{i} (j-1)_l 2^{-l} + \sum_{l=i+1}^{\infty} r_l 2^{-l}. \end{aligned}$$

Now if $0 < r \in I_0^i$ then $\tau(r) > i$ and so $Tr \in I_{2^i-1}^i$. Furthermore, if $r \in I_{2^i-1}^i$ then $r_1 = \ldots = r_i = 1$, and thus conclude that $(T^{-1}r)_1 = \ldots = (T^{-1}r)_i = 0$, that is, $T^{-1}r \in I_0^i$. Let $r \in [0,1)$ and $n \geq 1$ be arbitrary. Then $r \in I_j^n$ for some $0 \leq j \leq 2^n - 1$. For all $j - (2^n - 1) \leq k \leq j$,

$$T^k r = \sum_{l=1}^{n} (j-k)_l 2^{-l} + \sum_{l=n+1}^{\infty} r_l 2^{-l}. \tag{5}$$

Now since $T^{-1}I_j^i = I_{j+1}^i$ for $i \geq 1$, $j = 0, \ldots, 2^i - 2$, and the union over i and j of these sets generate the Borel σ-algebra, we conclude that T is measurable. Similar reasoning shows that T^{-1} is also measurable. The dynamical system $(\Omega, \mathcal{F}, \mu, T)$ is identified with $\Omega = [0,1)$ and $\mathcal{F}$ the Borel σ-algebra on $[0,1)$, T being the transformation developed above. Take μ to be Lebesgue measure on the unit interval. Since transformation T is measure-preserving on each set in the collection $\{I_j^i : 1 \leq j \leq 2^i - 1, 1 \leq i < \infty\}$ and these intervals generate the Borel σ-algebra $\mathcal{F}$, T is a stationary transformation. Now we prove that transformation T is ergodic as well. Assume $TA = A$. If $r \in A$ then $T^l r \in A$ for $-\infty < l < \infty$. Let $R_n : [0,1) \to \{0,1\}$ be the function $R_n(r) = r_n$. If r is chosen uniformly on $[0,1)$ then $R_1, R_2, \ldots$ is a series if i.i.d. random variables. Let $\mathcal{F}_n = \sigma(R_n, R_{n+1}, \ldots)$. By (5) it is immediate that $A \in \cap_{n=1}^{\infty} \mathcal{F}_n$ and so A is a tail event. By Kolmogorov's zero one law $\mu(A)$ is either zero or one. Hence T is ergodic.

Step 3. We define a partition of $[0,1)$ in the following way. Let $A_0 = \emptyset$, $B_0 = [0, 2^{-2})$, $C_0 = A_0 \bigcup B_0$. In general, for $i \geq 1$ let

$$A_i = \bigcup_{j=0}^{2^{i-1}-1} T^{-j}[0, 2^{-2i-1}) = \bigcup_{j=0}^{2^{i-1}-1} I_j^{2i+1} \tag{6}$$

and

$$B_i = \bigcup_{j=2^{i-1}}^{2^i-1} T^{-j}[0, 2^{-2i-1}) = \bigcup_{j=2^{i-1}}^{2^i-1} I_j^{2i+1}. \tag{7}$$

We show that

$$\mu(A_i) = \mu(B_i) = 2^{-i-2}.$$

Since $[0, 2^{-2i-1}) = I_0^{2i+1}$ it is clear for $0 \leq j < 2^i$ the sets $T^{-j}[0, 2^{-2i-1})$ are disjoint. Thus

$$\mu(A_i) = \mu(B_i) = 2^{-2i-1} 2^{i-1}.$$

Step 4. Put

$$C_i = A_i \bigcup B_i. \tag{8}$$

and

$$C = \bigcup_{j=0}^{\infty} C_i.$$

We show that

$$\mu(C) \le \frac{1}{2}.$$

Since $A_0 = \emptyset$ and for $i \ge 1$,

$$\begin{aligned}
A_i &= \bigcup_{j=0}^{2^{i-1}-1} T^{-j}[0, 2^{-2i-1}) \\
&= \left(\bigcup_{j=0}^{2^{i-1-1}-1} T^{-j}[0, 2^{-2i-1}) \right) \bigcup \left(\bigcup_{j=2^{i-1-1}}^{2^{i-1}-1} T^{j}[0, 2^{-2i-1}) \right) \\
&\subseteq \left(\bigcup_{j=0}^{2^{i-1-1}-1} T^{-j}[0, 2^{-2(i-1)-1}) \right) \bigcup \left(\bigcup_{j=2^{i-1-1}}^{2^{i-1}-1} T^{j}[0, 2^{-2(i-1)-1}) \right) \\
&= A_{i-1} \bigcup B_{i-1} \\
&= C_{i-1}
\end{aligned}$$

and so $A_i \subseteq C_{i-1}$, that is,

$$C = \bigcup_{i=0}^{\infty} B_i.$$

Furthermore

$$\begin{aligned}
\mu(C) &\le \sum_{i=0}^{\infty} \mu(B_i) \\
&= \sum_{i=0}^{\infty} 2^{-2i-1} 2^{i-1} \\
&= \sum_{i=0}^{\infty} 2^{-i-2} \\
&= \frac{1}{2}.
\end{aligned}$$

Step 5. Define the binary time series $\{Y_i\}$ as

$$Y_i(\omega) = \begin{cases} 1 & \text{if } T^i\omega \in C \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\{Y_i\}$ is stationary and ergodic since the dynamical system itself was so. $EY_i \le 0.5$ since $\mu(C) \le 0.5$. If $\omega \in A_i$ then by (6), (7) and (8)

$$\omega, T^{-1}\omega, \ldots, T^{-(2^{i-1}-1)}\omega \in C$$

that is

$$Y_0(\omega) = 1, \ldots, Y_{-(n_i-1)}(\omega) = 1.$$

Furthermore for $i > 16$, $\frac{n_i}{4} = \frac{2^{i-1}}{4} > \frac{8i^2}{4} = q_i$ and so $n_i = \frac{3}{4}n_i + \frac{1}{4}n_i > \frac{3}{4}n_i + q_i$. Thus

$$\mathbf{P}(\sum_{j=0}^{n_i-1} Y_{-j} > \frac{3}{4}n_i + q_i) \geq \mu(A_i).$$

By Step 3, for $i > 16$,

$$\mu(A_i) = 2^{-i-2} > 2^{-2i} = 2^{-(1/i)2i^2} = 2^{-\delta_i q_i}.$$

The proof of Proposition 1 is complete.

One could define

$$\lambda(\theta) := \limsup_{n\to\infty} \frac{1}{n} \log \mathbf{E} e^{\theta \sum_{i=1}^{n} Y_i}.$$

The next poposition shows that for the stationary and ergodic time-series just defined,

$$\{\theta; \lambda(\theta) - \theta s < 0\}$$

is the emty set when $s = 1$.

Proposition 2 *For $\{Y_i\}$ defined in the proof of Proposition 1,*

$$\lim_{i\to\infty} \frac{1}{n_i} \log \mathbf{E} e^{\sum_{j=0}^{n_i-1} \theta Y_j} = \theta.$$

Proof. By Step 3 and 5 of the proof of Proposition 1

$$\begin{aligned}
\theta &\geq \limsup_{i\to\infty} \frac{1}{n_i} \log \mathbf{E} e^{\sum_{j=0}^{n_i-1} \theta Y_j} \\
&\geq \liminf_{i\to\infty} \frac{1}{n_i \log_2 e} \log_2 \mathbf{E} 2^{\sum_{j=0}^{n_i-1} (\log_2 e)\theta Y_j} \\
&\geq \liminf_{i\to\infty} \frac{1}{n_i \log_2 e} \log_2 (2^{(\log_2 e)\theta n_i} \mu(A_i)) \\
&= \liminf_{i\to\infty} \theta + \frac{1}{n_i \log_2 e} \log_2 \mu(A_i) \\
&= \theta + \lim_{i\to\infty} \frac{2^{-i+1}}{\log_2 e} \log 2^{-i-2} = \theta.
\end{aligned}$$

The proof of Proposition 2 is complete.

References

[1] Dembo, A. and Zeitouni, O. (1992). *Large Deviations Techniques and Applications.* Jones and Bartlett Publishers.

[2] N. G. Duffield (1994) "Exponential bounds for queues with Markovian arrivals", *Queueing Systems*, 17, 413-430.

[3] N. G. Duffield, J. T. Lewis, N. O'Connel, R. Russel, F. Foomey. (1995) "Entropy of ATM traffic streams: a tool for estimating QoS parameters" *IEEE J. Selected Areas in Communications*, 13, 981-989.

[4] N. G. Duffield, N. O'Connel (1995) "Large deviations and overflow probabilities for the general single-server queue, with applications", *Proc. Cambridge Philos. Soc.*, 118, 363-374.

[5] P. W. Glynn, W. Whitt (1994) "Logarithmic asymptotics for steady-state tail probabilities in a single-server queue", *J. Applied Probability*, 118, 363-374.

[6] L. Györfi, G. Morvai, S. Yakowitz (1998) "Limits to consistent on-line forecasting for ergodic time series", *IEEE Trans. Information Theory*, 44, 886-892.

[7] D. V. Lindley (1952) "The theory of queues with a single server" *Proc. Cambridge Philos. Soc.*, 48, 277-289.

[8] T. Lindvall. (1992) *Lectures on the Coupling Method.* Wiley, New York.

[9] R. M. Loynes. (1962) "The stability of a queue with non-independent inter-arrival and service times", *Proc. Cambridge Philos. Soc.*, 58, 497-520.

[10] W.F. Stout. (1974) *Almost sure convergence.* Academic Press, New York.